

FAMILIES OF PERIODIC JACOBI-PERRON ALGORITHMS FOR ALL PERIOD LENGTHS

PAUL VOUTIER

ABSTRACT. For all integers $m \geq n \geq 2$, we exhibit infinite families of purely periodic Jacobi-Perron Algorithm (JPA) expansions of dimension n with period length equal to m along with the associated Hasse-Bernstein units. Some observations on the units of Levesque-Rhin as well as the periodicity of the JPA expansion of $(m^{1/3}, m^{2/3})$ are also made.

1. INTRODUCTION

The Jacobi-Perron Algorithm (JPA) is one of the generalisations of the ordinary continued fraction algorithm to higher dimensions. It has been the subject of considerable study (see, for example, [5, 15]).

Let $\boldsymbol{\alpha}^{(0)} = (\alpha_1^{(0)}, \dots, \alpha_{n-1}^{(0)}) \in \mathbb{R}^{n-1}$ for $n \geq 2$. The JPA expansion of $\boldsymbol{\alpha}^{(0)}$ is the sequence of elements of \mathbb{R}^{n-1} , $\langle \boldsymbol{\alpha}^{(\nu)} \rangle_{\nu \geq 0}$, defined by

$$\begin{aligned} \mathbf{a}^{(\nu)} &= (a_1^{(\nu)}, \dots, a_{n-1}^{(\nu)}) \\ &= (\lfloor \alpha_1^{(\nu)} \rfloor, \dots, \lfloor \alpha_{n-1}^{(\nu)} \rfloor), \\ \boldsymbol{\alpha}^{(\nu+1)} &= (\alpha_1^{(\nu+1)}, \dots, \alpha_{n-2}^{(\nu+1)}, \alpha_{n-1}^{(\nu+1)}) \\ &= \left(\frac{\alpha_2^{(\nu)} - a_2^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}}, \dots, \frac{\alpha_{n-1}^{(\nu)} - a_{n-1}^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}}, \frac{1}{\alpha_1^{(\nu)} - a_1^{(\nu)}} \right), \end{aligned}$$

where $\alpha_1^{(\nu)} \neq a_1^{(\nu)}$ and $\lfloor \cdot \rfloor$ is the greatest integer function.

For $n = 2$, this is the ordinary continued fraction algorithm. When $n = 3$, this algorithm was given by Jacobi [9] and is sometimes called the *Jacobi Algorithm*. Perron [14] extended Jacobi's definition to arbitrary n . As a result, for all $n \geq 2$, the algorithm is now known as the *Jacobi-Perron Algorithm*. Following [5, 11], we have presented it here in its inhomogeneous form.

The JPA expansion of $\alpha^{(0)}$ is called *periodic*, if there exist two integers ℓ_0, ℓ_1 with $\ell_0 \geq 0$ and $\ell_1 \geq 1$ such that

$$\alpha^{(\nu+\ell_1)} = \alpha^{(\nu)} \text{ for } \nu = \ell_0, \ell_0 + 1, \dots$$

If ℓ_0 and ℓ_1 are the smallest integers satisfying these conditions, then

$$\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\ell_0-1)} \text{ and } \alpha^{(\ell_0)}, \alpha^{(\ell_0+1)}, \dots, \alpha^{(\ell_0+\ell_1-1)}$$

are called respectively the *preperiod* and the *period* of the periodic JPA expansion, and ℓ_0 and ℓ_1 are their respective *lengths*. If $\ell_0 = 0$, then the JPA expansion of $\alpha^{(0)}$ is said to be *purely periodic*.

When the JPA expansion is periodic and all the $\alpha_i^{(0)}$'s are algebraic integers, then Hasse and Bernstein [6] proved that

$$\epsilon = \prod_{i=\ell_0}^{\ell_0+\ell_1-1} \alpha_{n-1}^{(i)}$$

is a unit in the ring of algebraic integers of $\mathbb{Q}(\alpha_1^{(0)}, \alpha_2^{(0)}, \dots, \alpha_{n-1}^{(0)})$. This is the *Hasse-Bernstein formula* and ϵ is called the *Hasse-Bernstein unit*.

Lagrange showed that the ordinary continued fraction expansion of a real number α is periodic if and only if α is a quadratic irrational. A central problem in the study of multidimensional generalisations of the ordinary continued fraction expansion is to find, and prove, an analogue of this result. The following question is part of JPA folklore.

Question. *When $1, \alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}$ form a \mathbb{Q} -basis of a number field of degree n , is the JPA expansion of $(\alpha_1, \dots, \alpha_{n-1})$ periodic?*

Numerical evidence suggests this is not always true for $n \geq 3$. There are many such tuples for which we do not know if the JPA expansion is periodic. For example, with $(\alpha_1, \alpha_2) = (m^{1/3}, m^{2/3})$ our own calculations (see [16]) show that $\ell_0 + \ell_1 > 30,000$ for many $4 \leq m \leq 5000$, including $m = 4, 6, 11, 13, 15, 19, 20$. In contrast, for all such JPA expansions known to be periodic with $m \leq 5000$, the maximum value of $\ell_0 + \ell_1$ is 93, occurring for $m = 17$.

Furthermore, also in the forthcoming [16], we present results suggesting structure to the periodic behaviour of such pure cubic (and n -th root) JPA expansions as well as supporting evidence for the following conjecture.

Conjecture 1. *Let $m = x^3 - x$ where $x \geq 2$ is a positive integer. The JPA expansion of $(m^{1/3}, m^{2/3})$ is not periodic.*

$x^3 - x$ appears to be one of many expressions of the form $ax^3 + bx$ with $a, b \in \mathbb{Z}$ for which this conjecture is true.

When $n = 3$ (i.e., the cubic case), only a few infinite classes of periodic Jacobi Algorithm expansions were known and only for period length at most 9, until 1991 when Levesque and Rhin [11] produced infinite classes for all period lengths of the form $3m + 1$ and $4m + 1$. They also determined the associated Hasse-Bernstein units, ω . Adam [2] proved that these ω are fundamental units in the ring of integers, $\mathbb{Z}[\omega]$.

In Section 5.2, we provide examples answering negatively the questions of Levesque and Rhin [11] on whether their Hasse-Bernstein units are fundamental units in the full ring of integers of $\mathbb{Q}(\omega)$.

For $n \geq 4$, less is known. Bernstein [3, 4] (see also [10, 12]) showed that one can construct purely periodic JPA expansions of period lengths 1, n and n^2 from $(D^n \pm d)^{1/n}$ when $n \geq 3$, $d|D$ and D is sufficiently large relative to d . In 1984, Paysant-Le Roux and Dubois [13] showed that every real number field, \mathbb{K} , of degree $n+1$ contains infinitely many n -tuples $(\alpha_1, \dots, \alpha_n)$ such that the JPA expansion of $(\alpha_1, \dots, \alpha_n)$ is purely periodic of length $n+1$ and $\{1, \alpha_1, \dots, \alpha_n\}$ is a \mathbb{Q} -basis for \mathbb{K} . Also, in her thesis, Adam [1] generalised the results of Levesque and Rhin to all $n \geq 2$ and all periods of the form $kn + 1$, where $n \leq k \leq 2n - 2$.

2. RESULTS

Here we extend the results of Levesque and Rhin to all dimensions, $n \geq 2$, and all period lengths, $m \geq n$, producing infinite families of expansions for each such m and n . In our expansions, the terms grow linearly for a fixed period length, m , rather than polynomially of degree m as in their work.

We start by defining n related recurrence sequences, $\{u_{0,m}\}_{m \geq 0}$, $\dots, \{u_{n-1,m}\}_{m \geq 0}$, for any n -tuple $(c_0, c_1, \dots, c_{n-1})$ of non-negative integers with $c_0, c_{n-1} \geq 1$. Set $u_{i,m} = \delta_{i,m}$, the Kronecker delta function, for $0 \leq i, m \leq n-1$, and define

$$(1) \quad u_{i,m} = c_{n-1}u_{i,m-1} + c_{n-2}u_{i,m-2} + \dots + c_0u_{i,m-n}$$

for $i = 0, 1, \dots, n-1$ and all $m \geq n$.

Theorem 1. *For all integers $c_0, c_1, \dots, c_{n-1}, m, n, t$ with $c_{n-1} \geq c_0 = 1$, $c_{n-1} \geq c_{n-2} \geq \dots \geq c_1 \geq 0$, $m \geq n \geq 2$, and $t \geq 1$, let*

$$f(X) = X^n - (u_{n-1,m}t + c_{n-1})X^{n-1} - \dots - (u_{0,m}t + c_0)$$

and let ω be the unique positive real root of $f(X)$. The JPA expansion of $\alpha^{(0)}$ with

$$(2) \quad \alpha_i^{(0)} = \sum_{j=0}^i (u_{j,m}t + c_j) \omega^{j-i}$$

for $i = 1, \dots, n-1$, is purely periodic of period length m and its $\mathbf{a}^{(\nu)}$'s are as in Table 1.

| $\nu \bmod m$ | $\mathbf{a}^{(\nu)}$ |
|---------------------|--|
| 0 | $(u_{1,m}t + c_1, \dots, u_{n-1,m}t + c_{n-1})$ |
| $1, \dots, m-n+1$ | (c_1, \dots, c_{n-1}) |
| $m-n+2, \dots, m-1$ | $(c_1, \dots, c_{m-\nu}, u_{m-\nu+1,m}t + c_{m-\nu+1}, \dots, u_{n-1,m}t + c_{n-1})$ |

TABLE 1. $\mathbf{a}^{(\nu)}$'s

Furthermore, the Hasse-Bernstein unit associated with this JPA expansion is

$$\epsilon = u_{n-1,m}\omega^{n-1} + u_{n-2,m}\omega^{n-2} + \dots + u_{1,m}\omega + u_{0,m}.$$

Remark 1. Applying $f(\omega) = 0$ to the expression in (2), we can also write $\alpha^{(0)}$ as

$$\left(\omega^{n-1} - \sum_{i=1}^{n-2} (u_{i+1,m}t + c_{i+1}) \omega^i, \dots, \omega^2 - (u_{n-1,m}t + c_{n-1}) \omega, \omega \right).$$

This choice of our starting vector follows the advice and experience of Dubois and Paysant-Le Roux with finding periodic JPA expansions [7, 8, 12], and in particular purely periodic ones.

Remark 2. If $c_0 > 1$, then we get a purely periodic expansion of period length $\text{lcm}(m, n)$ provided that $c_0 | c_i$ for all $i = 1, \dots, n-1$. It consists of $\text{lcm}(m, n)/m$ repetitions of the above pattern of $\mathbf{a}^{(\nu)}$'s but with c_0 in some of the denominators.

If c_0 does not divide all the other c_i 's, then computational evidence suggests that the expansion is not periodic.

3. PRELIMINARY LEMMAS

Lemma 2. *Let $n \geq 2$ be an integer and*

$$g(X) = X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \dots - a_0 \in \mathbb{R}[X]$$

with $a_{n-1} \geq 1$, $a_{n-1} \geq a_0 > 0$ and $a_{n-1} \geq a_{n-2} \geq \dots \geq a_1 \geq 0$. Then $g(X)$ has exactly one non-negative real root, ω ; it satisfies $a_{n-1} < \omega <$

$a_{n-1} + 1$. For all $1 \leq i \leq n-1$, we have

$$(3) \quad 0 < \sum_{j=0}^{i-1} a_j \omega^{j-i} < 1.$$

Proof. We first establish the statement about the non-negative real root of $g(X)$ and its location.

If $0 \leq x \leq a_{n-1}$, then $g(x) \leq -a_0 < 0$. From our condition on the coefficients of $g(X)$,

$$\begin{aligned} \frac{g(a_{n-1} + 1)}{(a_{n-1} + 1)^{n-2}} &\geq (a_{n-1} + 1)^2 - a_{n-1}(a_{n-1} + 1) - a_{n-1} \sum_{i=0}^{n-2} (a_{n-1} + 1)^{-i} \\ &> (a_{n-1} + 1)^2 - a_{n-1}(a_{n-1} + 1) - 2a_{n-1} = a_{n-1} - 1 \geq 0, \end{aligned}$$

since $a_{n-1} \geq 1$. Therefore, $g(a_{n-1} + 1) \geq 0$.

As $g'(x) > n(g(x) + a_0)/x$ for $x > 0$, so $g'(x) > 0$ for $x \geq \omega$. Hence there is only one non-negative real root of $g(X)$.

Since $a_0, \omega > 0$, the lower bound in (3) holds.

The upper bound in (3) is equivalent to $\sum_{j=0}^{i-1} a_j \omega^j < \omega^i$. We know from $g(\omega) = 0$, $a_0 > 0$, $a_{n-1} \geq \dots \geq a_1$ and $i \leq n-1$, that

$$\omega^n > \sum_{j=0}^{i-1} a_{n+j-i} \omega^{n+j-i} \geq \sum_{j=0}^{i-1} a_j \omega^{n+j-i}.$$

Dividing by ω^n completes the proof. \square

We also need relations between the elements of the recurrence sequences defined at the start of Section 2. Let c_0, c_1, \dots, c_{n-1} and $\{u_{0,m}\}_{m \geq 0}, \dots, \{u_{n-1,m}\}_{m \geq 0}$ be as there.

Lemma 3. (a) For $m \geq 1$,

$$(4) \quad u_{0,m} = c_0 u_{n-1,m-1}$$

and for $i = 1, \dots, n-1$,

$$(5) \quad u_{i,m} = c_i u_{n-1,m-1} + u_{i-1,m-1}.$$

(b) For $i = 1, \dots, n-1$ and $m \geq i+1$,

$$u_{i,m} = \sum_{j=0}^i c_j u_{n-1,m-i+j-1}.$$

Proof. (a) These relationships are immediately verified for $m \leq n$ using the initial conditions in the definitions of the sequences and observing that $u_{i,n} = c_i$. Applying the recurrence relation (1) and induction completes the proof for $m > n$.

(b) We prove this relation using part (a) and induction.

For $i = 1$,

$$u_{1,m} = c_1 u_{n-1,m-1} + u_{0,m-1} = c_1 u_{n-1,m-1} + c_0 u_{n-1,m-2},$$

as required.

Now consider $2 \leq i \leq n-1$. From (5), we have $u_{i,m} = c_i u_{n-1,m-1} + u_{i-1,m-1}$ and the desired relation now follows immediately by induction. \square

Finally, we need some inequalities satisfied by elements of these recurrence sequences too.

Lemma 4. (a) For $m \geq n$, $u_{i,m} \geq u_{i,m-1}$ for $i = 0, \dots, n-1$ and for at least one such i , $u_{i,m} > u_{i,m-1}$.

(b) For $m \geq 0$, $u_{i,m} \geq 1$ for at least one $i = 0, \dots, n-1$.

(c) Let $m \geq n-1$. Then $u_{i,m} \geq u_{i-1,m}$ for $2 \leq i \leq n-1$ and $u_{n-1,m} \geq u_{0,m}$.

(d) Suppose that $c_0 = 1$ and $c_{n-1} \geq c_{n-2} \geq \dots \geq c_1$. For $m \geq n$ and $1 \leq j \leq n-1$,

$$(6) \quad c_{j-1} u_{i,m-1} + \dots + c_1 u_{i,m-j+1} + c_0 u_{i,m-j} \leq u_{i,m} + 1$$

for all $0 \leq i \leq n-1$. For each $1 \leq j \leq n-1$, there exists $i_1 \in \{1, \dots, n-1\}$ such that $c_{j-1} u_{i_1,m-1} + \dots + c_1 u_{i_1,m-j+1} + c_0 u_{i_1,m-j} < u_{i_1,m}$, and there exists such an i_1 larger than any i for which equality holds in (6).

Proof. (a) The first statement follows by the recurrence relation (1), since $c_{n-1} \geq 1$.

Since $c_0 \geq 1$, for $n \leq m \leq 2n-1$, we have

$$u_{m-n,m} \geq c_{n-1} u_{m-n,m-1} + c_0 u_{m-n,m-n} \geq u_{m-n,m-1} + 1.$$

For $m \geq 2n$, we can take $i = 0$ and apply the recurrence relation (1), since $u_{0,n}, \dots, u_{0,2n-1} \geq 1$. This establishes the second statement in part (a).

(b) For $m \geq n$, this is immediate from (a). For $0 \leq m \leq n-1$, it follows from the initial values for the $u_{i,m}$'s.

(c) Applying first (5) with $i = n-1$ and then (4) from Lemma 3(a), we have $u_{n-1,m} = (c_{n-1}/c_0) u_{0,m} + u_{n-2,m-1} \geq u_{0,m}$ by our assumption that $c_{n-1} \geq c_0$ and since the $u_{i,j}$'s are non-negative. So we need only consider the first part of (c) in what follows.

This is true for $m = n-1$, since $u_{n-1,n-1} = 1$ and $u_{i,n-1} = 0$ for $i = 0, \dots, n-2$.

Let m be an integer with $n \leq m \leq 2n-1$ and suppose that (c) holds for $n-1 \leq j < m$. We can write

$$u_{i,m} = \begin{cases} c_{n-1}u_{i,m-1} + \cdots + c_{2n-m}u_{i,n} + c_{n-m+i} & \text{if } i \geq m-n \\ c_{n-1}u_{i,m-1} + \cdots + c_{2n-m}u_{i,n} & \text{otherwise,} \end{cases}$$

since $c_{2n-m}u_{i,n} = c_{n-(m-n)}u_{i,m-(m-n)}$ and $u_{j,k} = \delta_{j,k}$ for $0 \leq j, k \leq n-1$, by the definition of these recurrence sequences.

Part (c) now follows for m by our assumption that it holds for $n-1 \leq j < m$ and our assumption on the sizes of the c_i 's.

For $m \geq 2n$, our desired result follows immediately from the fact that it holds for $n-1 \leq m \leq 2n-1$ and the recurrence relation (1) defining the sequences.

(d) We start by showing that the stronger inequality

$$(7) \quad c_{j-1}u_{i,m-1} + \cdots + c_1u_{i,m-j+1} + u_{i,m-j} \leq u_{i,m},$$

holds for $m \geq n$ if $c_{n-j} \geq 1$ and for $m-j \geq n-1$ otherwise (this condition is best possible: with $c_0 = c_{n-1} = 1$ and $c_1 = \cdots = c_{n-2} = 0$, $i = n-2$, $j = n-1$ and $m = 2n-3$, we find that $u_{i,m} = 0$, while $c_{j-1}u_{i,m-1} + \cdots + c_1u_{i,m-j+1} + u_{i,m-j} = 1$).

For $m \geq n$, using the recurrence relation (1), to establish (7) we need to show that

$$(8) \quad \begin{aligned} & c_{j-1}u_{i,m-1} + \cdots + c_1u_{i,m-j+1} + u_{i,m-j} \\ & \leq c_{n-1}u_{i,m-1} + \cdots + c_{n-j+1}u_{i,m-j+1} + c_{n-j}u_{i,m-j} + \cdots + u_{i,m-n}, \end{aligned}$$

for all $1 \leq i \leq n-1$. Since $c_1 \leq \cdots \leq c_{n-1}$, this will follow provided $c_{n-j} \geq 1$ (or if $u_{i,m-j} = 0$, but we will not use that here).

Suppose that $c_{n-j} = 0$ and let k be the largest positive integer less than j such that $c_{n-k} \geq 1$. Since $j \leq n-1$, we have $c_{j-k} = 0$ and provided that $u_{i,m-k} \geq u_{i,m-j}$, (7) follows. Moreover, $u_{i,m-k} \geq u_{i,m-j}$ holds for $m-j \geq n-1$ by part (a).

Thus (7) holds under the stated conditions that follow it.

We now turn to proving that (6) holds. We need only consider the case when $m-j < n-1$ and $c_{n-j} = 0$. But in this case, $u_{i,m-j} \leq 1$, while the right-hand side of (8) is greater than or equal to the sum of the first $j-1$ terms on the left-hand side. Hence (6) follows.

Next, we show the existence of i_1 .

By part (b), $u_{i,m-n} \geq 1$ for at least one i for $m \geq n$, so under the conditions above (i.e., $m \geq n$ if $c_{n-j} \geq 1$ and $m-j \geq n-1$ otherwise), there is at least one i for which we have strict inequality in (7) too. The case when $m-j < n-1$ will be covered in the next paragraph.

It only remains to show that if we have equality in (6) for i , then we have strict inequality in (7) for some $i_1 > i$. From our analysis above

for when (7) holds, we know that we can only have equality in (6) if $m - j < n - 1$ and $c_{n-j} = 0$. So this can only occur if $u_{i,m-j} = 1$ (i.e., $i = m - j$). We take $i_1 = \min(m - k, n - 1) > m - j = i$, so

$$\begin{aligned} & c_{j-1}u_{i_1,m-1} + \cdots + c_{j-k}u_{i_1,m-k} + \cdots + c_1u_{i_1,m-j+1} + u_{i_1,m-j} \\ \leq & c_{n-1}u_{i_1,m-1} + \cdots + c_{n-k}u_{i_1,m-k} = u_{i_1,m}, \end{aligned}$$

since $u_{i_1,m-j} = 0$, $c_{n-1} \geq \cdots \geq c_1$ and $c_{n-k-1} = \cdots = c_1 = 0$.

Notice that $u_{i_1,m-k} \geq 1$, by our choice of i_1 . By our choice of k , we have $c_{n-k} = 1$ and $c_{j-k} = 0$. Thus $c_{n-k}u_{i_1,m-k} \geq 1$ and $c_{j-k}u_{i_1,m-k} = 0$. Therefore, we have strict inequality and for a value of i_1 satisfying the required conditions. \square

4. PROOF OF THEOREM 1

4.1. **$\mathbf{a}^{(0)}$.** From (2), we have $\alpha_i^{(0)} - (u_{i,m}t + c_i) = \sum_{j=0}^{i-1} (u_{j,m}t + c_j) \omega^{j-i}$. From Lemma 4(c) and our assumptions in Theorem 1 on the size of the c_j 's, we can apply Lemma 2 with $g(X) = f(X)$ and $a_i = u_{i,m}t + c_i$. From (4) in Lemma 2, the above sum for $\alpha_i^{(0)} - (u_{i,m}t + c_i)$ is strictly between 0 and 1 and so

$$(9) \quad \mathbf{a}^{(0)} = (u_{1,m}t + c_1, \dots, u_{n-1,m}t + c_{n-1}).$$

4.2. **Expression for $\alpha^{(1)}$ independent of t .** A crucial fact for us about $\alpha^{(1)}$ is that it can be written independently of t . We establish such an expression now.

From (2) and the above expression in (9) for $\mathbf{a}^{(0)}$, we obtain

$$(10) \quad \alpha_i^{(0)} - a_i^{(0)} = \omega^{-i} \sum_{j=0}^{i-1} (u_{j,m}t + c_j) \omega^j.$$

For $i = 1, \dots, n - 1$, we consider

$$\left(\alpha_i^{(0)} - a_i^{(0)} \right) (u_{n-1,m}\omega^{n-1} + \cdots + u_{0,m}).$$

Applying (10), we have

$$\begin{aligned}
& \left(\alpha_i^{(0)} - a_i^{(0)} \right) (u_{n-1,m} \omega^{n-1} + \cdots + u_{0,m}) \\
&= \omega^{-i} \left(\sum_{j=0}^{i-1} (u_{j,m} t + c_j) \omega^j \right) (u_{n-1,m} \omega^{n-1} + \cdots + u_{0,m}) \\
&= \omega^{-i} \left\{ t (u_{n-1,m} \omega^{n-1} + \cdots + u_{0,m}) \left(\sum_{j=0}^{i-1} u_{j,m} \omega^j \right) \right. \\
&\quad \left. + (u_{n-1,m} \omega^{n-1} + \cdots + u_{0,m}) \left(\sum_{j=0}^{i-1} c_j \omega^j \right) \right\}.
\end{aligned}$$

Since $f(\omega) = 0$, we can write this last expression as

$$\begin{aligned}
& \omega^{-i} \left\{ \left(\omega^n - c_{n-1} \omega^{n-1} - \cdots - c_i \omega^i - \left(\sum_{j=0}^{i-1} c_j \omega^j \right) \right) \left(\sum_{j=0}^{i-1} u_{j,m} \omega^j \right) \right. \\
&\quad \left. + \left(u_{n-1,m} \omega^{n-1} + \cdots + u_{i,m} \omega^i + \sum_{j=0}^{i-1} u_{j,m} \omega^j \right) \left(\sum_{j=0}^{i-1} c_j \omega^j \right) \right\} \\
&= \left(\omega^{n-i} - \sum_{j=i}^{n-1} c_j \omega^{j-i} \right) \left(\sum_{j=0}^{i-1} u_{j,m} \omega^j \right) + \left(\sum_{j=i}^{n-1} u_{j,m} \omega^{j-i} \right) \left(\sum_{j=0}^{i-1} c_j \omega^j \right).
\end{aligned}$$

We now show that for $i = 1, \dots, n-1$,

$$\begin{aligned}
(11) \quad & \left(\omega^{n-i} - \sum_{j=i}^{n-1} c_j \omega^{j-i} \right) \left(\sum_{j=0}^{i-1} u_{j,m} \omega^j \right) + \left(\sum_{j=i}^{n-1} u_{j,m} \omega^{j-i} \right) \left(\sum_{j=0}^{i-1} c_j \omega^j \right) \\
&= \sum_{k=0}^{n-1} \left(\sum_{j=0}^{i-1} c_j u_{k,m-i+j} \right) \omega^k.
\end{aligned}$$

From this, along with the relationship between $\alpha^{(0)}$, $\mathbf{a}^{(0)}$ and $\alpha^{(1)}$ that defines a JPA, it follows that for $i = 1, \dots, n-1$,

$$(12) \quad \alpha_i^{(1)} = \frac{\sum_{k=0}^{n-1} \left(\sum_{j=0}^i c_j u_{k,m-1-i+j} \right) \omega^k}{c_0 (u_{n-1,m-1} \omega^{n-1} + \cdots + u_{0,m-1})}.$$

This is immediate for $i = 1, \dots, n-2$. For $i = n-1$, we just need to apply (1) to the coefficient of each ω^k term in the numerator to establish the equality.

We prove our simplification by induction. Applying Lemma 3(a), for $i = 1$, the left-hand side of (11) is $u_{n-1,m-1} \omega^{n-1} + \cdots + u_{0,m-1}$, as required.

We now assume that (11) holds for some $1 \leq i \leq n-1$ and use some notation to assist our argument. Define

$$\begin{aligned} I_1^{(i)} &= \omega^{n-i} - \sum_{j=i}^{n-1} c_j \omega^{j-i}, & I_2^{(i)} &= \sum_{j=0}^{i-1} u_{j,m} \omega^j, \\ I_3^{(i)} &= \sum_{j=i}^{n-1} u_{j,m} \omega^{j-i} & \text{and} & \quad I_4^{(i)} = \sum_{j=0}^{i-1} c_j \omega^j. \end{aligned}$$

The left-hand side of (11) can be written as $I_1^{(i)} I_2^{(i)} + I_3^{(i)} I_4^{(i)}$ and we have

$$\begin{aligned} I_1^{(i+1)} &= \frac{I_1^{(i)} + c_i}{\omega}, & I_2^{(i+1)} &= I_2^{(i)} + u_{i,m} \omega^i, \\ I_3^{(i+1)} &= \frac{I_3^{(i)} - u_{i,m}}{\omega} & \text{and} & \quad I_4^{(i+1)} = I_4^{(i)} + c_i \omega^i. \end{aligned}$$

Expanding these expressions, we find that

$$\begin{aligned} & \omega \left(I_1^{(i+1)} I_2^{(i+1)} + I_3^{(i+1)} I_4^{(i+1)} \right) \\ &= \left(I_1^{(i)} I_2^{(i)} + I_3^{(i)} I_4^{(i)} \right) + u_{i,m} \omega^n + \sum_{k=0}^{n-1} (c_i u_{k,m} - c_k u_{i,m}) \omega^k. \end{aligned}$$

The desired expression for the left-hand side of this equation can be written as

$$(13) \quad \left(\sum_{j=0}^i c_j u_{n-1, m-i+j-1} \right) \omega^n + \sum_{k=1}^{n-1} \left(\sum_{j=0}^i c_j u_{k-1, m-i+j-1} \right) \omega^k,$$

while the right-hand side is

$$(14) \quad u_{i,m} \omega^n + \sum_{k=1}^{n-1} \left(-c_k u_{i,m} + \sum_{j=0}^i c_j u_{k, m-i+j-1} \right) \omega^k - c_0 u_{i,m} + \sum_{j=0}^i c_j u_{0, m-i+j}.$$

Using Lemma 3(b), we see that the coefficients of ω^n are equal. Similarly, using (4) from Lemma 3(a) along with Lemma 3(b), the constant coefficient in (14) is zero.

Finally, we compare the coefficients for $k = 1, \dots, n-1$. Using (4), we find that the coefficient of ω^k in (14) minus its counterpart in (13) is

$$-c_k u_{i,m} + c_k \sum_{j=0}^i c_j u_{n-1, m-i+j-1},$$

which is zero by again applying Lemma 3.

This completes our proof of (11) and thus (12).

4.3. Induction from $\alpha^{(\nu)}$ to $\alpha^{(\nu+1)}$: $1 \leq \nu \leq m - n$.

At this point, we use our assumption that $c_0 = 1$, the reason being that some of the $\alpha_i^{(\nu)}$'s below are divided by c_0 , depending on the values of both i and ν . This complicates the expressions in this subsection and the next one as well as resulting in a longer period, as stated above after Theorem 1.

Suppose that for some ν satisfying $1 \leq \nu \leq m - n$ and for all $i = 1, \dots, n - 1$, we have

$$\alpha_i^{(\nu)} = \frac{\sum_{k=0}^{n-1} \left(c_i u_{k,m-\nu} + c_0 u_{k,m-\nu-i} + \sum_{j=1}^{i-1} c_j u_{k,m-\nu-i+j} \right) \omega^k}{u_{n-1,m-\nu} \omega^{n-1} + \dots + u_{0,m-\nu}}.$$

From (12), this holds for $\nu = 1$, allowing us to proceed by induction. Thus

$$\alpha_i^{(\nu)} - c_i = \frac{\sum_{k=0}^{n-1} \left(c_0 u_{k,m-\nu-i} + \sum_{j=1}^{i-1} c_j u_{k,m-\nu-i+j} \right) \omega^k}{u_{n-1,m-\nu} \omega^{n-1} + \dots + u_{0,m-\nu}}.$$

By Lemma 4(b), for each $i = 0, \dots, n - 1$, there exists at least one $k \in \{0, \dots, n - 1\}$ such that $u_{k,m-\nu-i} \geq 1$. Thus both the denominator and numerator of this expression are positive for all such ν and so $\alpha_i^{(\nu)} - c_i > 0$.

From Lemma 2,

$$(15) \quad \omega > u_{n-1,m} t + c_{n-1} \geq (t + 1) c_{n-1} \geq 2 c_{n-1},$$

since $m \geq n$, $t \geq 1$ and since $u_{n-1,m} \geq u_{n-1,n} = c_{n-1}$ by Lemma 4(a).

By Lemma 4(d), we see that the coefficients of each ω^k in the numerator of the expression above for $\alpha_i^{(\nu)} - c_i$ is at most one more than the coefficient of the corresponding ω^k term in the denominator. Furthermore, Lemma 4(d) also allows us to show that even in the worst case (when $i_1 = n - 1$ and we have equality in (6) for all $0 \leq i \leq n - 2$)

$$\alpha_i^{(\nu)} - c_i \leq 1 + \frac{-\omega^{n-1} + \sum_{k=0}^{n-2} \omega^k}{u_{n-1,m-\nu} \omega^{n-1} + \dots + u_{0,m-\nu}}.$$

So by (15), $\alpha_i^{(\nu)} - c_i < 1$.

Using these expressions, we can write

$$\frac{\alpha_i^{(\nu)} - a_i^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}} = \frac{\sum_{k=0}^{n-1} \left(u_{k,m-\nu-i} + \sum_{j=1}^{i-1} c_j u_{k,m-\nu-i+j} \right) \omega^k}{\sum_{k=0}^{n-1} u_{k,m-\nu-1} \omega^k} = \alpha_{i-1}^{(\nu+1)},$$

for $i = 2, \dots, n - 1$.

Lastly,

$$\alpha_{n-1}^{(\nu+1)} = \frac{1}{\alpha_1^{(\nu)} - a_1^{(\nu)}} = \frac{u_{n-1,m-\nu}\omega^{n-1} + \cdots + u_{0,m-\nu}}{\sum_{k=0}^{n-1} u_{k,m-(\nu+1)}\omega^k},$$

from our above expression for $\alpha_i^{(\nu)} - c_i$. The desired expression for $\alpha_{n-1}^{(\nu+1)}$ holds from the recurrence relation (1) defining the $u_{k,m-\nu}$'s applied to the numerator.

4.4. Induction from $\alpha^{(\nu)}$ to $\alpha^{(\nu+1)}$: $m - n + 1 \leq \nu \leq m - 1$.

From our previous induction, we know that for $\nu = m - n + 1$,

$$\alpha_i^{(\nu)} = \begin{cases} \omega^{-i} + \sum_{j=1}^i c_j \omega^{j-i} & \text{if } 1 \leq i \leq m - \nu, \\ \omega^{n-i} - \sum_{j=i+1}^{n-1} (u_j t + c_j) \omega^{j-i} & \text{if } m - \nu + 1 \leq i \leq n - 1. \end{cases}$$

Now suppose for some $m - n + 1 \leq \nu \leq m - 1$, this holds. We proceed to determine $\mathbf{a}^{(\nu)}$ and $\alpha^{(\nu+1)}$.

From (15),

$$0 < \omega^{-i} + \sum_{j=1}^{i-1} c_j \omega^{j-i} < 1,$$

and so $a_i^{(\nu)} = c_i$ for $1 \leq i \leq m - \nu$. Thus, from Lemma 2, we have

$$\mathbf{a}^{(\nu)} = (c_1, \dots, c_{m-\nu}, u_{m-\nu+1,m}t + c_{m-\nu+1}, \dots, u_{n-1,m}t + c_{n-1}).$$

As before, we first consider $\alpha_1^{(\nu)} - a_1^{(\nu)} = 1/\omega$. We then verify that the above expressions for $\alpha_i^{(\nu)}$ hold for $\alpha_i^{(\nu+1)}$ as well.

Therefore, the expressions above also hold for $\alpha_i^{(m)}$, which we see equals $\alpha_i^{(0)}$, establishing our claim that this Jacobi-Perron expansion is periodic with preperiod length 0 and period length m .

4.5. The Hasse-Bernstein unit. From the expressions for the $\alpha_i^{(0)}$'s in Remark 1, they are algebraic integers and so we can find the Hasse-Bernstein unit from the JPA expansion. To do so, note that

$$\begin{aligned} \epsilon &= \prod_{\nu=0}^{m-1} \alpha_{n-1}^{(\nu)} = \omega^{n-1} \prod_{\nu=1}^{m-n+1} \frac{u_{n-1,m+1-\nu}\omega^{n-1} + \cdots + u_{0,m+1-\nu}}{u_{n-1,m-\nu}\omega^{n-1} + \cdots + u_{0,m-\nu}} \\ &= \omega^{n-1} \frac{u_{n-1,m}\omega^{n-1} + \cdots + u_{0,m}}{u_{n-1,n-1}\omega^{n-1} + \cdots + u_{0,n-1}} = u_{n-1,m}\omega^{n-1} + \cdots + u_{0,m}, \end{aligned}$$

as stated.

5. REMARKS ON HASSE-BERNSTEIN UNITS

5.1. Hasse-Bernstein units for $n = 3$. For $n = 3$, $f(X)$ in Theorem 1 has negative discriminant if and only if $c_2 \geq \lceil c_1^2/4 \rceil$. In this case, the unit group in the ring of integers of $\mathbb{Q}(\omega)$ is of rank 1. Our calculations in PARI suggest that, like the units in [11], the Hasse-Bernstein unit is a fundamental unit in the ring $\mathbb{Z}[\omega]$.

When we consider the full ring of integers of $\mathbb{Q}(\omega)$, it appears that the Hasse-Bernstein unit is often a fundamental unit of the unit group. However, it appears that for each tuple $(c_0 = 1, c_1, c_2)$ and each $m \geq 3$ satisfying the conditions of Theorem 1, there are infinitely many positive integers t such that the index is 2.

E.g., for $m = 6$ with $(c_0, c_1, c_2) = (1, 0, 1)$, define the sequence: $a_0 = -8, a_1 = 33$ and $a_k = -4a_{k-1} - a_{k-2} + 4$ for $k \geq 2$. If $t = 12a_k^2 - a_{k-1}^2 - 14a_k + 2a_{k-1} + 2$ for $k \geq 1$, then the group generated by the Hasse-Bernstein unit has index 2 in the full unit group.

Also for $m = 3$ with $(c_0, c_1, c_2) = (1, 0, 1)$ and $t = k^3 - 1$ with $k \geq 2$, the index is 3.

5.2. Hasse-Bernstein units from [11]. Something similar happens with the Hasse-Bernstein units in [11].

For example, with their family for their $4m + 1$ where $m = 1$, we define the sequence: $a_0 = 0, a_1 = -3$ and $a_k = 6a_{k-1} - a_{k-2} - 2$ for $k \geq 2$. If $c = 4a_k^2 - a_k a_{k-1} - 6a_k + a_{k-1} + 2$, then the group generated by the Hasse-Bernstein unit has index 2 in the full unit group.

Also with their family for their $4m + 1$ where $c = 3$ and their m is odd, the group generated by the Hasse-Bernstein unit has index 2.

For their family for their $3m + 1$, we give here just two examples. With $c = 2$ and $c = 779$, both with their $m = 1$, the groups generated by the Hasse-Bernstein unit have index 3 and 5 respectively.

These examples answer negatively the questions in [11] about their Hasse-Bernstein units being fundamental units in $\mathbb{Q}(\omega)$.

6. REMARKS ON $f(X)$ IN THEOREM 1

6.1. Irreducibility of $f(X)$. We have not shown that the polynomials, $f(X)$, in Theorem 1 are irreducible. Extensive computation suggests that they are, as well as the following more general conjecture.

Conjecture 2. For $n \geq 2$, let

$$g(X) = X^n - a_{n-1}X^{n-1} - \cdots - a_0 \in \mathbb{Z}[X],$$

where $a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq 0$ and $a_{n-1} \geq a_0 \geq 1$. Either

(a) if $a_{n-1} = a_{n-2}$, then $g(X)$ may be divisible by Φ_m , the cyclotomic

- polynomial of order m where $m|n$ and $1 < m < n$,
 (b) if n is even, then $g(X)$ may be divisible by $\Phi_2(X)$,
 (c) if $n \equiv 5 \pmod{6}$, then $g(X)$ may be divisible by $\Phi_6(X)$, or
 (d) $g(X)$ is irreducible in $\mathbb{Z}[X]$.

6.2. Roots of $f(X)$. In most other work on periodic JPA expansions, $\alpha^{(0)}$ is associated with either an n -th root of a rational number or a Pisot number. Here, in many cases, ω is a Pisot number, but that is not always the case. This is another novel aspect to our examples.

For example, if $n = 4$, $(c_0, c_1, c_2, c_3) = (1, 0, 1, 1)$, $m = 4$ and $t = 1$, then $f(X) = X^4 - 2X^3 - 2X^2 - 2$ also has a root at $x = -1.13418\dots$

So there exist polynomials satisfying the conditions in Conjecture 2 with a real root less than -1 when n is even. Such roots appear to be bounded below by the negative real root of $x^{n-1} + x^{n-2} + 1$. However, for n odd, such polynomials appear to have no real roots less than -1 .

There can also be non-real roots of $f(X)$ outside the unit circle. E.g., if $n = 4$, $(c_0, c_1, c_2, c_3) = (1, 0, 0, 1)$, $m = 5$ and $t = 10$, then $f(X) = X^4 - 11X^3 - 10X^2 - 11$ has a pair of non-real roots with absolute value $1.1908\dots$

ACKNOWLEDGEMENTS

I thank Claude Levesque for inspiring this work and for his stimulating conversations on this subject, as well as the referee for their thoughtful report and careful reading of the manuscript. Both played valuable, and much appreciated, roles in this work, its results and its presentation.

REFERENCES

1. B. Adam, *Développements Périodiques de Familles Paramétrées de Nombres Algébriques, Application à la recherche d'unités*, Thèse, Univ. de Metz, 1995.
2. B. Adam, *Voronoi-algorithm expansion of two families with period length going to infinity*, Math. Comp. 64 (1995), 1687–1704.
3. L. Bernstein, *Periodical Continued Fractions for Irrationals of Degree n by Jacobi's Algorithm*, J. Reine Angew. Math. 213 (1964), 31–38.
4. L. Bernstein, *New Infinite Classes of Periodic Jacobi-Perron Algorithms*, Pacific J. Math. 16 (1965), 1–31.
5. L. Bernstein, *The Jacobi-Perron algorithm – Its theory and application*, Lecture Notes in Mathematics 207, Springer-Verlag, Berlin, New York, 1971.
6. L. Bernstein, H. Hasse, *Einheitenberechnung mittels des Jacobi-Perronschen Algorithmus*, J. Reine Angew. Math. 218 (1965), 51–69.
7. E. Dubois, R. Paysant-Le Roux, *Développement périodique par l'algorithme de Jacobi-Perron et nombre de Pisot-Vijayaraghavan*, C. R. Acad. Sci. Paris, Sér. A Math. 272 (1971), 649–652.

8. E. Dubois, R. Paysant-Le Roux, *Algorithme de Jacobi-Perron dans les extensions cubiques*, C. R. Acad. Sci. Paris, Sér. A Math. 280 (1975), 183–186.
9. C.G.J. Jacobi, *Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welche jede Zahl aus drei vorhergehenden gebildet wird*, J. Reine Angew. Math. 69 (1868), 29–64.
10. C. Levesque, *A Class of Periodic Jacobi-Perron Algorithms in Pure Algebraic Number Fields of Degree $n \geq 3$* , Manuscripta Math. 22 (1977), 235–270.
11. C. Levesque, G. Rhin, *Two Families of Periodic Jacobi Algorithms with Period Lengths Going to Infinity*, J. Number Theory 37 (1991), 173–180.
12. R. Paysant-Le Roux, E. Dubois, *De nouvelles classes d'entiers pour lesquelles on connaît des développements périodiques par l'algorithme de Jacobi-Perron dans $\mathbb{Q}(m^{1/(n+1)})$* , C. R. Acad. Sci. Paris, Sér. A Math. 280 (1975), 57–59.
13. R. Paysant-Le Roux, E. Dubois, *Une application des nombres de Pisot à l'algorithme de Jacobi-Perron*, Monatsh. Math. 98 (1984), 145–155.
14. O. Perron, *Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus*, Math. Ann. 64 (1907), 1–76.
15. F. Schweiger, *Multidimensional continued fractions*, Oxford University Press, Oxford, 2000.
16. P. Voutier, *Remarks on the periodicity of Jacobi-Perron Algorithms associated with cube roots*, (to appear).

Current address: London, UK

E-mail address: paul.voutier@gmail.com